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Global Optimization of Marginal Functions with Applications to Economic Equilibrium ⋆

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Abstract. We discuss the applicability of the cutting angle method to global minimization of marginal functions. The search of equilibrium prices in the exchange model can be reduced to the global minimization of certain functions, which include marginal functions. This problem has been approximately solved by the cutting angle method. Results of numerical experiments are presented and discussed.

Key words: Global optimization, Marginal function, Cutting angle method, Economic equilibrium, Exchange model

1. Introduction

Marginal functions, that is, functions of the form

$$\psi(p) = \max_{y \in a(p)} f(p, y), \tag{1.1}$$

where a is a set-valued mapping, have been intensively studied by many researches (see, for example, [11, 14, 16] and references therein).

Note, that the so-called max function:

$$\psi(p) = \max_{y \in Y} f(p, y) \tag{1.2}$$

is the simplest example of a marginal function. Here Y can be considered as the image of a constant mapping a. We assume that the set Y in (1.2) is finite dimensional, infinite and compact.

Marginal functions arise in the study of many problems of mathematical economics.

Global minimization of marginal functions (in particular, max functions) is a very complicated problem. Indeed, almost all known methods of global minimization require to compute values of the objective function many times. However, the value of marginal function ψ of (1.1) at a point p can be found only by solving the optimization problem

$$f(p, y) \longrightarrow \max \text{ subject to } y \in a(p),$$

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so the calculation of the value of ψ is very time consuming.

Recently the so-called cutting angle method for global optimization of Lipschitz function has been developed (see [2, 6, 7, 23]). We need to compute only a few values of the objective function at each iteration of the cutting angle method, so we can hope that this method is suitable for minimizing some marginal functions.

In this paper we apply the cutting angle method to solving some problems of global optimization, which arise in the theory of economic equilibrium. The objective functions of these problems are not always Lipschitz, so we need to transform objective functions in order to obtain Lipschitz continuity. Note that the data in models of economic equilibrium often is not very precise. Thus, we can restrict ourselves to the search of *approximate solutions* of corresponding problems of global optimization. We show that the cutting angle method, which can find an approximate solution fairly quickly, can serve for solving these problems.

Consider a market economy with m consumers (agents) and n goods. The consumer j has a utility function U^j and a vector of initial endowments ω^j. It has been shown in [20] that, under some natural assumptions, a vector of equilibrium prices of this model can be found as a solution of the following problem:

$$H(p) \longrightarrow \min \text{ subject to } p \in \operatorname{ri} S = \left\{ p : p_i > 0, i = 1, \ldots, n; \sum_i p_i = 1 \right\}, \tag{1.3}$$

with $H(p) = H_1(p) - H_2(p)$ where $H_1(p)$ and $H_2(p)$ are special marginal functions (see Section 5). Problem (1.3) has been studied from various points of view in [1] and [5]. The similar approach for Arrow–Debreu equilibrium model was proposed in [21].

The equilibrium exists if and only if the value of problem (1.3) is equal to zero. It is well known that the equilibrium does exist if the utility function U^j is quasiconcave and ω^j is a strictly positive vector for all j. If these conditions hold, we need to solve problem (1.3) with the known value of global minimum. If at least one of these conditions is not valid, problem (1.3) can serve for recognition of the existence of the equilibrium.

The structure of this paper is as follows. In Section 2 we discuss problems, which arise under minimization of marginal functions. In Section 3 we recall briefly the cutting angle method and in Section 4 we recall the exchange model of economic equilibrium. Section 5 provides a discussion of a reformulation of an equilibrium problem as a special optimization problem. In Section 6, we study Lipschitz continuity of the objective function of this optimization problem. Section 7 provides a discussion of the calculation of equilibrium prices by the cutting angle method. We record results of numerical experiments in this section. Appendix contains data, which were used for numerical experiments.

2. Minimization of marginal functions

Let a be a set-valued mapping transforming a set P into the set of all non-empty subsets of a set Y. Consider a function $f : P \times Y \to \mathbb{R}$. A function ψ defined on P by

$$\psi(p) = \sup_{y \in a(p)} f(p, y) \tag{2.1}$$

is called a *marginal function*. In this paper we assume that P and Y are closed subsets of finite-dimensional spaces and images of the mapping a are compact sets. Assume also that the function $y \mapsto f(p, y)$ is upper semicontinuous for all $p \in P$. Then the supremum in (2.1) is attained. Continuity, Lipschitz continuity and directional differentiability of marginal functions have been intensively studied (see, for example, [4, 11, 14, 16] and references therein).

A well-known example of marginal functions is delivered by a parametric problem of mathematical programming:

$$f(p, y) \longrightarrow \max \ \text{ subject to } \ g_i(p, y) \leqslant 0 \ (i \in I), \ \ h_j(p, y) = 0 \ (j \in J).$$

Here $a(p) = \{y : g_i(p, y) \leqslant 0, \ (i \in I), \ h_j(p, y) = 0 \ (j \in J)\}$. It is assumed that the set $a(p)$ is nonempty and compact for all $p \in P$.

Marginal functions arise in the study of many problems of mathematical economics. Assume, for example, that a is a production mapping of a producer, that is, $a(x)$ is the set of all outputs, which can be produced by the producer from an input x. Let p be a price vector (that is a vector with positive coordinates). Then the maximal profit $\pi(x)$, which can be obtained by the producer, is equal to $(\max_{y \in a(x)}[p, y]) - [p, x]$, where $[u, v]$ is the inner product of vectors u and v. The marginal function $\max_{y \in a(x)}[p, y]$ describes the revenues of the producer. A different kind of marginal functions arises in the study of a consumer behaviour. Assume that a consumer has a utility function U. Then the maximal utility, which can be obtained by the consumer is

$$v(p) = \max_{y \in B(p)} U(y),$$

where $B(p)$ is the set of vectors of goods, which are available for the consumer, if there is a price vector p at the market.

Let $\phi(p) = g(\psi_1(p), \ldots, \psi_k(p))$, where $g : \mathbb{R}^k \to \mathbb{R}$ and $\psi_i, \ i = 1, \ldots, k$ are marginal functions and let P be a compact set. Consider the following problem of global optimization:

$$\phi(p) \longrightarrow \min \quad \text{subject to} \quad p \in P. \tag{2.2}$$

Many problems from various fields of mathematics and its applications can be represented in form (2.2). We mention here only bilevel programming (see for example [24] and references therein) and problems of economic equilibrium (see [20]).

Consider the simplest version of (2.2), namely the problem

$$\psi(p) \longrightarrow \min \ \text{ subject to } \ p \in P, \tag{2.3}$$

where ψ is a marginal function defined by (2.1). A very special case of problem (2.3), where the mapping a in (2.1) is constant: $a(p) = Y$ for all $p \in P$ is known as a minimax problem. Thus the minimax problem has a form: to find a point $(\tilde{p}, \tilde{y}) \in P \times Y$ such that

$$f(\tilde{p}, \tilde{y}) = \min_{p\in P} \max_{y\in Y} f(p, y). \tag{2.4}$$

If Y is finite, we have a discrete minimax problem, if Y is a compact infinite set, we have a continuous minimax problem.

Discrete minimax problems have been studied by many authors (see, for example [10, 17] and references therein). We consider here only continuous minimax problems, which are much more complicated.

We mention three numerical methods, which were proposed for solving continuous minimax problems in the seventies and earlier: Arrow–Hurwicz method (see [3, 15]), the net method (see [10]); and the method of extremal basis (see [9]). Modern approaches to solving these problems can be found in the book [17] by E. Polak. The main attention in this book is paid to the minimization of convex functions of the form $\psi(p) = \max_{y\in Y} f(p, y)$. The net method and the method of consistent approximation [17] can be applied for the search for local minima of ψ, when this function is non-convex. These methods are based on an approximation of the given set Y by a finite set Y'. Having such an approximation we can substitute a continuous minimax problem for a sequence of discrete minimax problems and then solve these problems by known algorithms (see, for example [10, 17]. These methods require to use large finite sets Y', hence we need to minimize the maximum of a large number of functions.

The calculation of local minima of a non-convex marginal function ψ is very time-consuming. Indeed, the application of a majority of known numerical methods for local optimization is based on the calculation of the objective function and its subgradients (in a certain sense) in many different points. Sometimes it is possible to use only approximate values of the function and its subgraients, however, in order to find an approximate value of the function ϕ at a point $\bar{p}$ we need to find an approximate solution of the problem

$$f(\bar{p}, y) \longrightarrow \max \ \text{ subject to } \ y \in a(\bar{p}). \tag{2.5}$$

Thus the search for a local minimum of the function ϕ requires to solve (approximately) problem (2.5) very often.

Note that as a rule the calculation of subgradients of a non-convex marginal function is more complicated than the calculation of a value of this function. It leads to the following conclusion: optimization methods, which are based on the calculation both functions and its subgradients, as a rule are not applicable for the local minimization of non-convex marginal functions. Methods, which are based only on calculation of values of the objective function (so-called derivative-free methods), are more preferable.

Without any convexity assumptions a marginal function may have a lot of local minimizers, so we need to use methods of global optimization for solving problem (2.2). However, the majority of these methods (branch-and-bound, random search, etc.) require very many objective function evaluations. So, these methods are not applicable for minimization of marginal functions (and even for solving minimax problems) if the dimension of the problem is sufficiently high. For global optimization of marginal functions we need to find methods, which require a small amount of objective functions evaluations. One of such methods is the so-called cutting angle method, which uses only few function evaluations at each iteration. So we can propose that the cutting angle method is applicable for the global minimization of marginal functions.

3. Cutting angle method

Let $I = \{1, \ldots, n\}$. Consider the space $\mathbb{R}^n$ of all vectors $(x_i)_{i \in I}$. We shall use the following notations.

- x_i is the i-th coordinate of a vector $x \in \mathbb{R}^n$;
- if $x, y \in \mathbb{R}^n$ then $x \geqslant y \iff x_i \geqslant y_i$ for all $i \in I$;
- if $x, y \in \mathbb{R}^n$ then $x \gg y \iff x_i > y_i$ for all $i \in I$;
- $\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x \geqslant 0\}$;
- $\mathbb{R}^n_{++} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x \gg 0\}$;
- $\mathbf{1} = (1, \ldots, 1)$;
- $[l, x] = \sum_{i=1}^n l_i x_i$ is the inner product of vectors l and x.

A function f defined on the cone $\mathbb{R}^n_+$ of all n-vectors with nonnegative coordinates is called an IPH function if f is increasing ($x \geqslant y \implies f(x) \geqslant f(y)$) and positively homogeneous of the first degree ($f(\lambda x) = \lambda f(x)$ for all $x \geqslant 0$ and $\lambda > 0$). The following result holds (see [23]).

THEOREM 3.1 *Let f be a Lipschitz function defined on the unit simplex $S = \{x \in \mathbb{R}^n_+ : \sum_{i \in I} x_i = 1\}$. Then there exists a constant $c > 0$ such that for all $c' \geqslant c$ the function $x \mapsto f(x) + c'$ $(x \in S)$ is the restriction of a certain IPH function g to S.*

Thus the minimization of a Lipschitz function f over S is reduced to the minimization of an IPH function g over S. The minimization of a Lipschitz function subject to linear constraints can be transformed to the minimization of another Lipschitz function over the simplex (see [23] for details), hence the minimization of a Lipschitz function subject to linear constraints is reduced to the minimization of an IPH function over the simplex.

The cutting angle method was proposed (see [2] and also [22]) for the minimization of a so-called ICAR (increasing convex-along-rays) function defined on $\mathbb{R}^n_+$ over a compact subset of $\mathbb{R}^n_+$. We consider here only a version of this method, which is suitable for the minimization of an IPH (increasing positively homogeneous of degree one) function over the unit simplex. This version has been proposed and discussed in detail in [6, 7]. It follows from monotonicity of an IPH function f that $f(x) \geqslant f(0) = 0$ for all $x \in \mathbb{R}^n_+$. We assume in the sequel that $f(x) > 0$ for all $x \neq 0$. For $x \in \mathbb{R}^n_+$ we shall use the following notation: $I(x) = \{i \in I : x_i > 0\}$, c/x is the vector with the following coordinates:

$$\left(\frac{c}{x}\right)_i = \begin{cases} \frac{c}{x_i} & \text{if } i \in I(x); \\ 0 & \text{if } i \notin I(x). \end{cases}$$

The cutting angle method is based on the following result (see [22] and references therein).

THEOREM 3.2 *(1) Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a function such that $f(x) > 0$ for all $x \neq 0$. Then f is IPH if and only if there exists $U \subset \mathbb{R}^n_+ \setminus \{0\}$ such that*

$$f(x) = \max_{l \in U} \min_{i \in I(l)} l_i x_i, \qquad x \in \mathbb{R}^n_+;$$

(2) Let $x^0 \in \mathbb{R}^n_+ \setminus \{0\}$ and $l = f(x^0)/x^0$. Then

$$\min_{i \in I(l)} l_i x_i \leqslant f(x) \textit{ for all } x \in \mathbb{R}^n_+ \quad \textit{and} \quad \min_{i \in I(l)} l_i x_i^0 = f(x^0).$$

Let $e^k = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the unit vector such that $I(e^k) = \{k\}$. Clearly $f(e^k)/e^k = f(e^k)e^k$ $(k = 1, \ldots, n)$.

First we present the simplest version of the cutting angle method.

The cutting angle method for minimizing an IPH function over the unit simplex

Step 0. (Initialization) Take points $x^k = e^k$, $k = 1, \ldots, n$. Let $l^k = f(x^k)/x^k$, $k = 1, \ldots, n$. Define the function h_n :

$$h_n(x) = \max_{k \leqslant n} \min_{i \in I(l^k)} l_i^k x_i = \max_{k \leqslant n} l_k^k x_k$$

and set $j = n$.

Step 1. Find a solution x^* of the problem

$$h_j(x) \longrightarrow \min \text{ subject to } x \in S. \tag{3.1}$$

Step 2. Set $j = j + 1$ and $x^j = x^*$.

Step 3. Compute $l^j = f(x^j)/x^j$, define the function

$$h_j(x) = \max(h_{j-1}(x), \min_{i \in I(l^j)} l_i^j x_i) \equiv \max_{k \leqslant j} \min_{i \in I(l^k)} l_i^k x_i$$

and go to Step 1.

The convergence of the cutting angle method has been proved under very mild assumptions (see [16], where the convergence of much a more general method was established, and also [22]).

REMARK 3.1 The solution of the auxiliary problem (3.1) is the most difficult part of the algorithm. The special method for the solution of this problem has been developed in [6, 7]. We do not discuss this method here. Different approaches are possible in lower dimensions ([2, 23]): in particular, the auxiliary problem can be reduced to a mixed-integer linear programming problem, which can be solved by standard optimization packages (for example, CPLEX).

REMARK 3.2 Only one value of the objective function f should be calculated at each iteration.

REMARK 3.3 A more advanced version of the cutting angle method was proposed in [7]. All approximate solutions of the problem (3.1) are considered in this version. We shall use the advanced version in this paper, since it allows us to fasten the search for a global minimizer (see [7] for details). The advanced version may require a few calculation of the objective function at each iteration.

REMARK 3.4 Let

$$\lambda_j = \min_{x \in S} h_j(x) = h_j(x^{j+1}).$$

It follows from Theorem 3.2 that $h_j(x) \leqslant f(x)$ for all $x \in S$. Hence

$$\lambda_j = \min_{x \in S} h_j(x) \leqslant \min_{x \in S} f(x).$$

Thus λ_j is a lower estimate of the global minimum of f over S. Let $\mu_j = f(x^j)$. It can be shown (see, for example [22] and references therein) that λ_j is an increasing sequence and $\lambda_j - \mu_j \to 0$ as $j \to +\infty$. So we have a stopping criterion, which enables us (at least theoretically) to obtain an approximate solution with an arbitrary given tolerance.

REMARK 3.5 The cutting angle method can be considered as a special case of many well-known algorithms (see [22] for a short survey of some of these algorithms). However, the numerical implementation of the cutting angle method has demonstrated that it works much better than many other versions of these algorithms.

4. The Equilibrium Model

We study the so-called exchange models of economic equilibrium. First we consider a *classical* version of this model. The classical exchange model describes a market, where n goods are circulated. Let $I = \{1, \dots, n\}$. There is a finite number, say m, of economical agents, which are called *consumers*, at the market. Let $J = \{1, \dots, m\}$. A consumer $j \in J$ is described by a pair (U^j, ω^j), where U^j is a utility function (objective function) of the consumer j and $\omega^j \in \mathbb{R}^n_+$ is a vector of her initial endowments.

A state of economy is a vector $X = (x^1, \dots, x^m) \in (\mathbb{R}^n_+)^m$. A state X is called feasible if it belongs to the set

$$\Omega = \{X = (x^j)_{j \in J} : \sum_{j \in J} x^j \leqslant \omega\}, \tag{4.1}$$

where

$$\omega = \sum_{j \in J} \omega^j \tag{4.2}$$

is the vector of all products available on the market.

A vector $p \in \mathbb{R}^n_+ \setminus \{0\}$ is called a *price vector*. Having a price vector p, the consumer j can sell her initial endowment ω^j and can buy a vector x^j from the set

$$B^j(p) = \{x \in \mathbb{R}^n_+ : [p, x] \leqslant [p, \omega^j]\}.$$

This set is called the budget set of the consumer j.

Note that $B^j(p) = B^j(\lambda p)$ for all $\lambda > 0$, so we can assume without loss of generality that the mapping B^j is defined only on the unit simplex $S = \{p \in \mathbb{R}^n_+ : \sum_{i=1}^n p_i = 1\}$. We shall denote the model under consideration by $\mathcal{M}$:

$$\mathcal{M} = \{I, J, (U^j)_{j \in J}, (\omega^j)_{j \in J}\}. \tag{4.3}$$

A semi-equilibrium of the model $\mathcal{M}$ is a pair $(\bar{p}, \bar{X})$, where $\bar{p}$ is a price vector and $X = (\bar{x}^j)_{j \in J}$ is a feasible state, such that $\bar{x}^j$ is a solution of the consumer problem:

$$U^j(x^j) \longrightarrow \max \text{ subject to } x^j \in B^j(\bar{p}). \tag{4.4}$$

The vector $\bar{p}$ is called *equilibrium prices* and the vector $\bar{x}^j$ is called an *equilibrium state* of the agent j.

It is well known that a semi-equilibrium exists if functions U^j are quasiconcave and vectors ω^j are strictly positive (see, for example, [13] and references therein). We assume in the rest of this paper that

$$\omega^j \gg 0, \qquad j \in J.$$

We shall consider both models with quasiconcave U^j and without quasiconcave U^j.

Let $(\bar{p}, \bar{X})$ be a semi-equilibrium of the model $\mathcal{M}$. Since $\bar{X}$ is a feasible state, the inequality $\sum_j \bar{x}^j \leqslant \omega$ holds, where ω is defined by (4.2). A semi-equilibrium $(\bar{p}, \bar{X})$ is called an *equilibrium* if $\sum_j \bar{x}^j = \omega$. It is well known (and easy to check), that each semi-equilibrium $(\bar{p}, \bar{X})$ with $\bar{p} \gg 0$ is an equilibrium.

We accept the following

ASSUMPTION 4.1 *There exists $j \in J$ such that the utility function U^j is non-satiated in the following sense: for each $x \in \mathbb{R}^n_+$ and each $i \in I$ there exists $\lambda_i > 0$ such that $U^j(x + \lambda_i e^i) > U^j(x)$, where e^i is the i-th unit vector.*

Then each vector of equilibrium prices $\bar{p}$ is strictly positive. Indeed, assume that $\bar{p}_i = 0$. Let $\bar{x}^j$ be an equilibrium state of the consumer j. Then the budget set $B^j(p) = \{x : [\bar{p}, x] \leqslant [\bar{p}, \omega^j]\}$ of this consumer contains the ray $\{\bar{x}^j + \lambda e^i : \lambda > 0\}$, hence there exists $x^j := \bar{x}^j + \lambda^i e^i \in B^j(p)$ such that $U^j(x^j) > U^j(\bar{x}^j)$, which contradicts the definition of the semi-equilibrium. Thus Assumption 4.1 allows us to consider only strictly positive equilibrium prices (therefore, an equilibrium is guaranteed to exist by the above).

We also consider one more type of exchange models, namely a model of exchange with fixed budgets. In contrast with the classical version, it is assumed that each agent j has a fixed budget, that is, a special sum of money d_j, which does not depend on market prices and ω^j. The budget mapping $B^j(p)$ should be replaced for the mapping

$$\tilde{B}^j(p) = \{x \in \mathbb{R}^n_+ : [p, x] \leqslant d_j\}, \qquad p \neq 0. \tag{4.5}$$

Having mappings $\tilde{B}^j(p)$, we can define the semi-equilibrium and equilibrium in this model in the same way as in the classical case. Note that in contrast with the classical model, $\tilde{B}^j(\lambda p) \neq \lambda \tilde{B}^j(p)$ for $\lambda > 0$ so we need to consider mappings $\tilde{B}^j$ on the cone $\mathbb{R}^n_+ \setminus \{0\}$. The existence results for the model with fixed budgets are similar to those for the classical model. If Assumption 4.1 holds then equilibrium prices are strictly positive.

REMARK 4.1 It is known [13, 18] that an equilibrium for a model with fixed budgets and positively homogeneous utility functions can be found by solving a convex programming problem. However this does not hold if utility functions are not positively homogeneous. In this paper we consider a search for an economical equilibrium as an example of application of the cutting angle method. From this point of view models with fixed budget is of a special interest since they can be reduced to a problem of global optimization, which is different from that for classical exchange models.

5. Equilibrium Prices as a Solution of a Special Optimization Problem

It is not hard to formulate a bilevel problem such that its solution set coincides with the set of equilibrium prices. Let Assumption 4.1 hold. Consider the set ri $S =$

$\{p \gg 0 : \sum_{i\in I} p_i = 1\} = S \cap \mathbb{R}^n_{++}$. Assume for the sake of simplicity that the utility functions U^j are strictly concave. Then the consumer's problem

$$U^j(x^j) \longrightarrow \max \text{ subject to } x^j \in B^j(p) \tag{5.1}$$

has a unique solution for each $p \in \operatorname{ri} S$. Denote this solution by $x^j(p)$. Consider a function

$$\gamma(p) = \left\| \sum_{j\in J} x^j(p) - \omega \right\|$$

Clearly $\gamma(p) \geqslant 0$ and $\gamma(p) = 0$ if and only if p is a vector of equilibrium prices. Thus equilibrium prices can be found as a solution of the following bilevel problem:

$$\left\| \sum_{j\in J} x^j(p) - \omega \right\| \longrightarrow \min \quad \text{subject to} \quad p \in \operatorname{ri} S \tag{5.2}$$

where $x^j(p)$ is a solution of problem (5.1). The optimal value of problem (5.2) is known (and equal to zero). However, this problem is very complicated. Note that the set-valued mapping B^j is not Lipschitz on $\operatorname{ri} S$. (See, for example, [4] for the definition of Lipschitz set-valued mappings.) So we cannot hope that the mappings $x^j(p)$ $(j \in J)$ and the function $\gamma(p)$ are Lipschitz for an arbitrary functions U^j. Hence we cannot guarantee that problem (5.2) can be solved by the cutting angle method.

We now consider a certain different type of optimization problems, which can serve for the search of economic equilibrium. Such problems were suggested in [20]. For each $p \in \operatorname{ri} S$ consider sets

$$A_*(p) = \{X = (x^j)_{j\in J} : x^j \in B^j(p) \ (j \in J)\} \equiv \prod_{j\in J} B^j(p) \tag{5.3}$$

and

$$A(p) = \{X = (x^j)_{j\in J} \in A_*(p) : \sum_{j\in J} x^j \leqslant \omega\}. \tag{5.4}$$

Both sets $A_*(p)$ and $A(p)$ are compact for $p \in \operatorname{ri} S$, so the following functions are well defined:

$$H_1(p) = \max_{X=(x^j)\in A_*(p)} \sum_{j\in J} U^j(x^j) = \sum_{j\in J} \max_{x^j\in B^j(p)} U^j(x^j), \qquad p \in \operatorname{ri} S \tag{5.5}$$

and

$$H_2(p) = \max_{X=(x^j)\in A(p)} \sum U^j(x^j), \qquad p \in \operatorname{ri} S. \tag{5.6}$$

Let $H(p) = H_1(p) - H_2(p)$. Clearly $H(p) \geqslant 0$. The following assertion holds (see [20]).

LEMMA 5.1 *Let Assumption 4.1 hold and let $\bar{p} \in \operatorname{ri} S$. The equality $H(\bar{p}) = 0$ is valid if and only if $\bar{p}$ is a vector of equilibrium prices.*

Thus in order to find a vector of equilibrium prices we need to solve the following problem of global optimization:

$$H(p) \longrightarrow \min \ \text{ subject to } \ p \in \operatorname{ri} S. \tag{5.7}$$

The objective function of this problem is the difference of two marginal functions H_1 and H_2. If the equilibrium exists then the minimal value of this problem is known and equal to zero.

We accept the following assumption.

ASSUMPTION 5.1

(1) *For each $j \in J$ there exists a point $a_j \ll 0$ such that the utility function U^j is twice continuously differentiable on the set $a_j + \mathbb{R}^n_{++}$ and*

$$[\nabla^2 U^j(x^j)y^j, y^j] < 0 \textit{ for all } x^j \in a_j + \mathbb{R}^n_{++} \textit{ and } y^j \neq 0. \tag{5.8}$$

(2) *The utility function U^j $(j \in J)$ is increasing in the following sense: if $x^1 \gg x^2$ then $U^j(x^1) > U^j(x^2)$.*

(3) $\lim_{x \to +\infty} U^j(x) = +\infty$ *for each $j \in J$.*

It follows from (5.8) that U^j is a strictly concave function. If Assumption 5.1 holds then for each $p \gg 0$ and each $j \in J$ the consumer's problem

$$U^j(x) \longrightarrow \max \ \text{ subject to } x \in B^j(p) \equiv \{x \geqslant 0 : [p, x] \leqslant [p, \omega^j]\} \tag{5.9}$$

has a unique solution $x^j(p)$ and $[p, x^j(p)] = [p, \omega^j]$. Hence, the problem

$$\sum_{j \in J} U^j(x^j) \longrightarrow \max \ \text{ subject to } \ X = (x^j)_{j \in J} \in A_*(p),$$

which serves for the definition of the function H_1, has a unique solution $X(p) = (x^j(p))_{j \in J}$. The problem

$$\sum_{j \in J} U^j(x^j) \longrightarrow \max \ \text{ subject to } \ X = (x^j)_{j \in J} \in A(p), \tag{5.10}$$

which serves for the definition of H_2 also has a unique solution $\tilde{X}(p) = (\tilde{x}^j(p))_{j \in J}$.

It has been proved in [21] that the function H_1 is Frechet differentiable with the piece-wise C^1 gradient mapping ∇H_1. We have

$$\nabla H_1(p) = \sum_j \frac{1}{p_{i_j}} \left(\frac{U^j(x^j(p))}{x_{i_j}} \right) (\omega^j - x^j(p)), \tag{5.11}$$

where i_j is an arbitrary index belonging to $I^j(p) := \{i' : x^j_{i'}(p) > 0\}$. The function H_2 is directionally differentiable and, under some additional assumptions, also Frechet differentiable with the piece-wise C^1 gradient mapping. Assume that there exist $j \in J$, $i_j \in I$ and a sequence $p_{(k)} \in \text{ri}\, S$ such that $i_j \in I^j(p_{(k)})$ for all k and $(p_{(k)})_{i_j} \to 0$ as $k \to +\infty$. It follows from (5.11) that

$$\|\nabla H_1(p)\| \to +\infty \text{ as } k \to +\infty.$$

So we cannot hope that $H(p)$ is a Lipschitz function on the set $\text{ri}\, S = \{p \in S : p \gg 0\}$ for all quasiconcave utility functions U^j and we cannot guarantee that the global minimizer of H over the simplex S can be found by the cutting angle method.

In order to get a Lipschitz function we shall slightly change the definition of the function H. First we note that the function H_2 is bounded on $\text{ri}\, S$. Indeed, let $p \in \text{ri}\, S$ and let $X = (x^j) \in A(p)$. Then $\sum_{j\in J} x^j \leqslant \omega$, hence $x^j \leqslant \omega$ for all $j \in J$. Since the utility functions U^j are increasing, we have

$$H_2(p) = \max_{X\in A(p)} \sum_{j\in J} U^j(x^j) \leqslant C,$$

where

$$C = \sum_{j\in J} U^j(\omega). \tag{5.12}$$

However the function H_1 can be unbounded on S. Since we are interested in the global minimum of the function $H = H_1 - H_2$ we can throw off points where the function H_1 is very large. For this purpose we consider the function

$$L(p) = \min(H(p), d), \tag{5.13}$$

where d is an arbitrary positive number, which is bigger than a global minimum of the function H. Clearly global minimizers of functions H and L coincide so if the equilibrium exists, then a point $p \in \text{ri}\, S$ is an equilibrium prices if and only if $L(p) = 0$.

Thus a vector of equilibrium prices can be found as a solution of the following problem:

$$L(p) \longrightarrow \min \text{ subject to } p \in \text{ri}\, S. \tag{5.14}$$

6. Lipschitz Continuity of the Function L.

In this section we shall check that the function L is Lipschitz. Let $E = \{p \in S : H(p) \leqslant d\}$. It is sufficient to verify that L is Lipschitz on the set E. First we prove the following simple Proposition.

PROPOSITION 6.1 *The function H_1 is bounded on the set E.*

Proof: If $p \in E$ then $H_1(p) \leqslant H_2(p)+d \leqslant C+d$ where C is defined by (5.12). □

COROLLARY 6.1 *There exists a vector $a \in \mathbb{R}^n_{++}$ such that $x^j(p) \leqslant a$ for all $p \in E$ and $j \in J$.*

Indeed, $U^j(x(p)) \leqslant H_1(p)$. So Proposition 6.1 and Assumption 5.1(3) imply the boundness of the set $\{x^j(p) : p \in E, j \in J\}$.

We need the following theorem which is a special case of a general result from [19]. (This special case and its relation with the mentioned general result were discussed in [21].)

THEOREM 6.1 *Let P_* be an open subset of a finite dimensional space and $p \in P_*$. Let $x_*(p)$ be a solution of the following parametric convex programming problem:*

$$T(x) \longrightarrow \max$$

subject to

$$x \in \mathbb{R}^n, \quad [a^k, x] \leqslant b^k \ (k \in K), \quad [p, A^r x - h^r] \leqslant 0 \ (r \in R), \tag{6.1}$$

where K and R are finite sets, a^k $(k \in K)$, h^r $(r \in R)$ are vectors and A^r $(r \in R)$ are matrices. Assume that T is a twice continuously differentiable concave function defined on an open set Ω such that

$$[\nabla^2 T(x_*(p))y, y] < 0 \text{ for all } y \neq 0.$$

Assume also that the Mangasarian-Fromovitz constraint qualification holds at the point $x_*(p)$, that is, there exists $y \in \mathbb{R}^n$ such that

$$[a_k, y] < 0 \text{ if } k \in K(x_*(p)) \text{ and } [p, A^r y] < 0 \text{ if } r \in R(x_*(p)),$$

where

$$K(x_*(p)) = \{k \in K : [a_k, x_*(p)] = b_k\},$$
$$R(x_*(p)) = \{r \in R : [p, A^r x_*(p) - h^r] = 0\}.$$

Then the function $x^*(p)$ is piece-wise continuously differentiable near the point p. In particular, $x^*(p)$ is a locally Lipschitz function near the point p.

In order to apply this theorem, we need to consider an open set $P_* \supset S$ such that $[p, \omega^j] > 0$ for all $p \in P_*$ and $j \in J$. Such a set exists since $\omega^j \gg 0$ for all $j \in J$. For $p \in P_*$ consider the following two optimization problems.
Problem P_1:

$$U^j(x) \longrightarrow \max \text{ subject to } [p, x] \leqslant [p, \omega^j], \ 0 \leqslant x \leqslant a,$$

where a is a vector from Corollary 6.1. Problem P_1 has a unique solution $x_*^j(p)$.
Problem P_2:

$$\sum_{j\in J} U^j(x^j) \longrightarrow \max \ \text{subject to} \ \sum_{j\in J} x^j \leqslant \omega,\ x^j \geqslant 0,\ [p, x^j] \leqslant [p, \omega^j]\,(j \in J)$$

This problem has a unique solution $\tilde{X}_*(p) = (\tilde{x}_*^j(p))_{j\in J}$.

PROPOSITION 6.2 *(1) For each $p \in P_*$ the Mangasarian–Fromovitz constraint qualification holds for problem P_1 at the point $x_*^j(p)$.*
(2)For each $p \in P_*$ the Mangasarian–Fromovitz constraint qualification holds for problem P_2 at the point $\tilde{X}_*(p)$.

We leave the simple but cumbersome proof of this proposition to the reader.

PROPOSITION 6.3 *Let*

$$H_*(p) = \sum_{j\in J} U^j(x_*^j(p)) - \sum_{j\in J} U^j(\tilde{x}_*^j(p)). \tag{6.2}$$

Then the function H_* is locally Lipschitz on the set P_*.

Proof: It follows from Theorem 6.1 and Proposition 6.2 a that functions $x_*^j(p)$ and $\tilde{x}_*^j(p)$ are locally Lipschitz. Functions U^j are locally Lipschitz as well. Hence $H_*(p)$ is locally Lipschitz. □

COROLLARY 6.2 *The function $H_*(p)$ is Lipschitz on the compact set S.*

THEOREM 6.2 *The function L defined by (5.13) is Lipschitz on the simplex S.*

Proof: It is sufficient to prove that the function H is Lipschitz on the set $E = \{p \in S : H(p) \leqslant d\}$. Since $x^j(p) \leqslant a$ for $p \in E$ (see Corollary 6.1), it follows that $x^j(p) = x_*^j(p)$ for $p \in E$. We also have $X^j(p) = X_*^j(p)$ for all $p \in S$. Hence $L(p) = H(p) = H_*(p)$ for $p \in E$. The result follows from Corollary 6.2. □

7. The Calculation of Equilibrium Prices by the Cutting Angle Method

In this section we discuss results of numerical experiments, which were carried out in order to find an approximate equilibrium prices or to verify that the equilibrium does not exist. The problem (5.14):

$$L(p) \longrightarrow \max \ \text{subject to} \ p \in \operatorname{ri} S$$

has been solved by the cutting angle method.

First we give some remarks.

1. We consider a search of an economical equilibrium as an example of applications of the cutting angle method to approximate global optimization of a complicated function, which is the difference of two marginal functions. Currently the developed approach can mainly be used for research purposes. We hope that the further development of the cutting angle method will allow one to use this approach for many real problems of economic equilibrium.

2. Assume that the equilibrium exists. We have $L(p) = \min(H(p), d)$, so actually we are looking for a global minimum of the function H, which is equal to zero. We also use the function H for determining the precision of results of calculations. This function is defined with the help of utility functions U^j, which describe the preferences of agents. Note that the function $\rho^j U^j$ with $\rho^j > 0$ and function U^j describes the same preferences (see [20] for a corresponding discussion). Thus if we replace the function H for the function λH with arbitrary λ, we obtained the same approximate global minimizers, however the measure of the precision will be changed. In order to avoid this situation, we need to consider only functions U^j, which are normalized in a certain sense.

All numerical experiments were carried out with utility functions of the form

$$(*) \quad U^j(x) = c^j \prod_{i=1}^{n} (x_i + b_i)^{\alpha_i^j}.$$

Since $H = H_1 - H_2$ and both H_1 and H_2 are defined by maximization of the sum $\sum_{j \in J} U^j(x)$, we consider the following normalization of function H:

$$\sum_{j \in J} c^j = 1. \tag{7.1}$$

3. The cutting angle method is suitable for the search of approximate solution of a global optimization problem. If there exists the equilibrium of the model under consideration, then the optimal value of the normalized function H is equal to zero. We search for vectors p such that $H(p) \approx 0.001 \div 0.003$. Assume that we consider a model, for which the existence of the equilibrium are not proved and we want to learn whether the equilibrium exists. Recall (see Section 3) that the cutting angle method produces the lower estimates λ_j of the global minimum. If $\lambda_j > 0$ for some j, then the equilibrium does not exist.

4. We used the exact penalty method for solving internal problems, which are problems of convex programming (maximization of a concave function subject to linear constraints). Thus, the exact penalty functions have been constructed; for their minimization we used the so-called discrete gradient method ([5]). Since the precision of the results obtained by the cutting angle method is approximately $0.001 \div 0.003$, internal problems were solved with the precision 10^{-4}.

5. We used a computer IBM Pentium-S CPU 150 MHz. Problems with $n = 4$ and $m = 8$ were mainly considered. It takes approximately 15–18 minutes to find a solution of such a problem with the precision $0.001 \div 0.003$. Results of numerical experiments show that the solution of internal problems takes the main part of CPU time.
Our aim is to show that the cutting angle method can be successfully applied for solving the problem (5.14). Using more effective methods of convex optimization for solving internal problems and more iterations of the cutting angle method we can find a more precise solution of problem (5.14).

(1) Classical exchange models with concave utility functions

First we consider a classical exchange model such that Assumption 4.1 and Assumption 5.1 hold. Then the equilibrium does exist, so the value of problem (5.14) is known and equal to zero. Since the function L is Lipschitz over the unit simplex, the cutting angle method can be applied.

EXAMPLE 7.1 Consider the economical system with 8 consumers and 4 goods. The utility functions are defined as follows:

$$U^j(x^j) = c_j \prod_{i=1}^{4} (x_i^j + b_i^j)^{\alpha_i^j}, \quad \sum_{i=1}^{4} \alpha_i^j = 0.8, \ \alpha_i^j \geqslant 0,$$
$$i = 1, \dots, 4, \ \ j = 1, \dots, 8. \tag{7.2}$$

Vectors $\alpha^j = (\alpha_1^j, \alpha_2^j, \alpha_3^j, \alpha_4^j)$ and $b^j = (b_1^j, b_2^j, b_3^j, b_4^j)$, $j = 1, \dots, 8$ are the rows of the matrices A_1 and B_1, respectively. Vectors ω^j, $j = 1, \dots, 8$ are rows of matrix Ω_1. Coefficients c_j, $j = 1, \dots, 8$ are coordinates of the vector c^1. Matrices A_1, B_1, Ω_1 and vector c^1 can be found in Appendix.

Numerical results for Example 7.1. The point $p = (0.3333, 0.3102, 0.0001, 0.3564)$ with $H(p) = 0.0018$ was found by the cutting angle method after 18 iterations. It takes 22 objective function evaluations.

EXAMPLE 7.2 We again consider the same economical system. Utility functions have the same form with the same vector of coefficients c^1 and the same matrix B_1. However coefficients α_i^j are defined in different way. We assume now that

$$\sum_{i=1}^{4} \alpha_i^j = 1, \ \alpha_i^j \geqslant 0, \ i = 1, \dots, 4, \ \ j = 1, \dots, 8.$$

Vectors $\alpha^j = (\alpha_1^j, \alpha_2^j, \alpha_3^j, \alpha_4^j)$, $j = 1, \dots, 8$ are the rows of the matrix A_1', which is in Appendix.

Numerical results for Example 7.2. The point $p = (0.2852, 0.3867, 0.3280, 0.0001)$ with $H(p) = 0.0026$ was found by the cutting angle method after 17 iterations. It takes 21 objective function evaluations.

(2) Exchange model with fixed budgets and concave utility functions

Consider a model with fixed budgets such that Assumption 4.1 and Assumption 5.1 hold. We now use the approach similar to that for the classical model. Let $p \in \mathbb{R}^n_{++}$. Consider set-valued mappings

$$\tilde{A}_*(p) = \{X = (x^j)_{j\in J} : x_j \in \tilde{B}^j(p),\ (j \in J)\},$$

$$\tilde{A}(p) = \left\{ X = (x^j)_{j\in J} \in \tilde{A}(p) : \sum_{j\in J} x^j \leqslant \omega \right\},$$

where $\tilde{B}^j$ defined by (4.5). Let $\tilde{H}(p) = \tilde{H}_1(p) - \tilde{H}_2(p),\ \ (p \in \mathbb{R}^n_{++})$, where

$$\tilde{H}_1(p) = \sum_{j\in J} \max_{x^j \in B^j} U^j(x^j), \qquad \tilde{H}_2(p) = \max_{X=(x_j)_{j\in J}\in A(p)} \sum_{j\in J} U^j(x^j),$$
$$(p \in \mathbb{R}^n_{++}).$$

Further, let $\tilde{L}(p) = \min(\tilde{H}(p), d)$, where d is an arbitrary positive number. The same argument as in Section 5 demonstrates that a vector of equilibrium prices can be found as a solution of the problem:

$$\tilde{L}(p) \longrightarrow \max \text{ subject to } p \in \mathbb{R}^n_{++}. \tag{7.3}$$

In order to apply the cutting angle method we need to transform this problem to a problem with a simplicial constraint. For this purpose we introduce constraints $p_i \leqslant M$ where M is a sufficiently large number. (It is assumed that there exist a vector $\bar{p}$ of equilibrium prices such that $\bar{p}_i \leqslant M$ for all $i \in I$.) Consider the set

$$D = \left\{ p \in \mathbb{R}^n_{++} : \sum_{i\in I} p_i \leqslant nM \right\}.$$

Clearly this set contains the set $\{p : p_i \leqslant M,\ i = 1, \dots, n\}$. Let $t = nM$. Adding a new variable p_{n+1} and replacing p_i/t by q_i we transform the set D to the unit simplex $S \subset \mathbb{R}^{n+1}_+$.

EXAMPLE 7.3 Consider the economical system of Example 7.2. It is assumed now that each consumer j has the fixed budget d_j and her budget set $\tilde{B}^j(p)$ has the following form

$$\tilde{B}^j(p) = \{x \in \mathbb{R}^n_+ : [p, x] \leqslant d_j\}, \quad j = 1, \dots, 8.$$

Vector $d^1 = (d_1, \dots, d_8)$ can be found in Appendix.

Numerical results for Example 7.3. The point $p = (0.3707, 0.0203, 0.2030, 0.2030)$ with $H(p) = 0.0015$ was found by the cutting angle method after 5 iterations. It takes 24 objective function evaluations. Note that $\sum_{i=1}^{4} p_i < 1$.

(3) Classical exchange model with not necessarily concave utility functions

The existence of the equilibrium in classical model can be proved if utility functions U^j are quasiconcave. If at least one of these functions is not quasiconcave we cannot guarantee that the equilibrium exists. However, we also cannot guarantee that the equilibrium does not exist. In order to check the existence of the equilibrium we can solve problem (5.14). If the value of this problem is equal to zero then the equilibrium exists and a global minimizer of the function L is a vector of equilibrium prices. Otherwise, the equilibrium does not exist. Some numerical experiments have been carried out in order to check the existence of equilibrium. We consider utility functions, which can be represented as the maximum of two concave functions. (Functions of such structure arise when indivisible goods are considered.)

We now describe numerical experiments which were carried out.

1. A number of models with two goods and two consumers has been examined. Each consumer has a utility functions, which is the maximum of two functions of the form (*). Coefficients c^j and b_i were chosen at random. Coefficients α_i^j were chosen also at random, however it is assumed that either $\sum_i \alpha_i^j = 1$ or $\sum_i \alpha_i^j = 0.8$. A model with three goods and three consumers, which have utility functions of the same form with chosen at random coefficients, also has been examined. Numerical experiments showed that all models under consideration possess equilibrium. These results allow us to consider the following conjecture: *if the number of goods is equal to the number of consumers and a utility function of each consumer is the maximum of two functions, for which Assumption 4.1 and Assumption 5.1 hold, then the classical exchange model has an equilibrium*. This conjecture was discussed with Professor J.-M. Bonnisseau, who suggested the following counter-example.

EXAMPLE 7.4 (J.-M. Bonnisseau, [8]). Consider a classical exchange model with two goods and two consumers. Utility functions U^j and vectors ω^j (j =1,2) of consumers have the following form:

$$U^1(x^1) = \sqrt{(x_1^1 + 1)(x_2^1 + 1)}, \qquad \omega^1 = (4, 4);$$

$$U^2(x^2) = \max((x_1^2 + 1)^{1/3}(x_2^2 + 1)^{2/3}, (x_1^2 + 1)^{2/3}(x_2^2 + 1)^{1/3}), \quad \omega^2 = (4.4).$$

The demand of the consumers can be explicitly calculated. This calculations showed that an equilibrium does not exist.

Numerical experiments, based on the approach suggested in this paper, also confirmed that the model under consideration and also some similar models have no equilibrium.

Nevertheless our numerical experiments which were performed by a random choice of coefficients show that there exists many models, which have the described above form and possess an equilibrium.

2 A model with two products and three consumers has been examined. Each consumer has a utility function, which is the maximum of two functions of the form (*) with chosen at random coefficients. This model has no equilibrium, however each its submodel with two consumers possesses an equilibrium.

We now present one of the corresponding examples.

EXAMPLE 7.5 Consider two economical systems, one of them has 3 consumers and 2 goods, the other has 3 consumers and 3 goods. Assume the consumer j has a utility function U^j:

$$U^j(x^j) = \max\{U^{j1}(x^j), U^{j2}(x^j)\}, \quad j = 1, 2, 3,$$

where

$$U^{jk}(x^j) = c_j^k \prod_{i=1}^{m} (x_i^j + b_i^{jk})^{\alpha_i^{jk}}, \quad m = 2 \text{ or } 3,$$

$$\sum_{i=1}^{m} \alpha_i^{jk} = 1, \quad \alpha_i^{jk} \geqslant 0, \ i = 1, \dots, m, \ j = 1, 2, 3, \ k = 1, 2.$$

For the first system vectors $\alpha^j = (\alpha_1^{j1}, \alpha_2^{j1}, \alpha_1^{j2}, \alpha_2^{j2})$, $b^j = (b_1^{j1}, b_2^{j1}, b_1^{j2}, b_2^{j2})$ and ω^j, $j = 1, 2, 3$ are the rows of matrices A_3, B_2 and Ω_2, respectively, which can be found in Appendix. Coefficients $(c_1^1, c_1^2, c_2^1, c_2^2, c_3^1, c_3^2)$ are coordinates of the vector c^2 (see Appendix).

For the second system vectors

$$\alpha^j = (\alpha_1^{j1}, \alpha_2^{j1}, \alpha_3^{j1}, \alpha_1^{j2}, \alpha_2^{j2}, \alpha_3^{j2}), \ b^j = (b_1^{j1}, b_2^{j1}, b_3^{j1}, b_1^{j2}, b_2^{j2}, b_3^{j2})$$

and w^j, $j = 1, 2, 3$ are the rows of matrices A_3, B_3 and Ω_3, respectively (see Appendix). Coefficients $(c_1^1, c_1^2, c_2^1, c_2^2, c_3^1, c_3^2)$ are coordinates of the vector c^2.

First we describe our approach for solving internal problems.

Note that the function H_1 has the form

$$\begin{aligned} H_1(p) &= \sum_{j \in J} \max_{x^j \in B^j(p)} U^j(x^j) = \sum_{j \in J} \max_{x^j \in B^j(p)} \max\{U^{j1}(x^j), U^{j2}(x^j)\} \\ &= \sum_{j \in J} \max \left\{ \max_{x^j \in B^j(p)} U^{j1}(x^j), \max_{x^j \in B^j(p)} U^{j2}(x^j) \right\}, \qquad p \in \operatorname{ri} S. \end{aligned}$$

Hence, we can find the value $H_1(p)$ of the function H_1 at a point p by solving the convex programming problems:

$$U^{ji}(x^j) \to \max,\; x^j \in B^j(p), \quad j = 1, 2, 3,\; i = 1, 2,$$

We also have for $p \in \operatorname{ri} S$:

$$\begin{aligned} H_2(p) &= \max_{X=(x^j)\in A(p)} \sum U^j(x^j) = \max_{X=(x^j)\in A(p)} \sum_{j=1}^{3} \max\{U^{j1}(x^j), U^{j2}(x^j)\} \\ &= \max_{i_1,i_2,i_3=1,2}\ \max_{X=(x^j)\in A(p)} (U^{1i_1}(x^1) + U^{2i_2}(x^2) + U^{3i_3}(x^3)). \end{aligned} \tag{7.4}$$

In order to find the value $H_2(p)$ of the function H_2 at the point p we consider all possible combinations:

$$\bar{U}(X) = U^{1i_1}(x^1) + U^{2i_2}(x^2) + U^{3i_3}(x^3), \quad i_1, i_2, i_3 = 1, 2.$$

Then we solve the convex programming problems :

$$\bar{U}(X) \to \max,\; X \in A(p),$$

and calculate $H_2(p)$ by (7.4).

Solving problems (5.14) for the described models, we found that the first economic system does not possesses the equilibrium. However, all its subsystems consisting of 2 consumers and 2 goods have the equilibrium. The second system also has the equilibrium.

8. Appendix

In this Appendix we report the data, which were used for numerical experiments.
Examples 8.1–8.3

$$A_1 = \begin{pmatrix} 0.05205 & 0.35366 & 0.24230 & 0.15199 \\ 0.23562 & 0.08699 & 0.30802 & 0.16937 \\ 0.25335 & 0.17358 & 0.10888 & 0.26420 \\ 0.09515 & 0.33689 & 0.18525 & 0.18271 \\ 0.21114 & 0.13245 & 0.32138 & 0.13503 \\ 0.25861 & 0.14220 & 0.14025 & 0.25893 \\ 0.13280 & 0.32224 & 0.13540 & 0.20955 \\ 0.18448 & 0.18196 & 0.33593 & 0.09763 \end{pmatrix}$$

$$A'_1 = \begin{pmatrix} 0.06506 & 0.44207 & 0.30288 & 0.18999 \\ 0.29452 & 0.10874 & 0.38502 & 0.21171 \\ 0.31668 & 0.21697 & 0.13610 & 0.33025 \\ 0.11893 & 0.42112 & 0.23156 & 0.22839 \\ 0.26393 & 0.16556 & 0.40172 & 0.16879 \\ 0.32326 & 0.17775 & 0.17532 & 0.32367 \\ 0.16601 & 0.40281 & 0.16925 & 0.26194 \\ 0.23060 & 0.22745 & 0.41991 & 0.12204 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0.28793 & 0.06658 & 0.23274 & 0.20954 \\ 0.24972 & 0.22317 & 0.10869 & 0.29905 \\ 0.12910 & 0.22351 & 0.19709 & 0.20271 \\ 0.28129 & 0.05614 & 0.27284 & 0.17613 \\ 0.26395 & 0.16693 & 0.16528 & 0.28286 \\ 0.09218 & 0.19242 & 0.20010 & 0.20564 \\ 0.25339 & 0.06827 & 0.29563 & 0.12479 \\ 0.25775 & 0.15915 & 0.20537 & 0.24766 \end{pmatrix}$$

$$\Omega_1 = \begin{pmatrix} 0.22732 & 0.17509 & 0.41652 & 0.27501 \\ 0.24565 & 0.18920 & 0.45010 & 0.29718 \\ 0.03812 & 0.02936 & 0.06985 & 0.04612 \\ 0.20445 & 0.15747 & 0.37461 & 0.24734 \\ 0.25905 & 0.19953 & 0.47466 & 0.31340 \\ 0.07548 & 0.05814 & 0.13831 & 0.09132 \\ 0.17749 & 0.13670 & 0.32521 & 0.21472 \\ 0.26728 & 0.20586 & 0.48973 & 0.32334 \end{pmatrix}$$

$$c^1 = (0.08402, 0.13197, 0.19386, 0.02938, 0.16941, 0.17038, 0.02761, 0.19336)$$

$$d^1 = (20.80734, 32.68218, 48.00851, 7.27500)$$

Example 8.4

$$A_2 = \begin{pmatrix} 0.12829 & 0.87171 & 0.73035 & 0.26965 \\ 0.59343 & 0.40657 & 0.22022 & 0.77978 \\ 0.61452 & 0.38548 & 0.64521 & 0.35479 \end{pmatrix}$$

$$B_2 = \begin{pmatrix} 0.28793 & 0.06658 & 0.24972 & 0.22317 \\ 0.12910 & 0.22351 & 0.28129 & 0.05614 \\ 0.26395 & 0.16693 & 0.09218 & 0.19242 \end{pmatrix}$$

$$\Omega_2 = \begin{pmatrix} 0.22732 & 0.17509 \\ 0.24565 & 0.18920 \\ 0.03812 & 0.02936 \end{pmatrix}$$

$$c^2 = (0.10786, 0.16941, 0.24885, 0.03771, 0.21747, 0.21870),$$

$$A_3 = \begin{pmatrix} 0.08032 & 0.54576 & 0.37392 & 0.37362 & 0.13794 & 0.48843 \\ 0.47284 & 0.32395 & 0.20321 & 0.15414 & 0.54576 & 0.30010 \\ 0.31752 & 0.19918 & 0.48330 & 0.47796 & 0.26282 & 0.25922 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0.28793 & 0.06658 & 0.23274 & 0.24972 & 0.22317 & 0.10869 \\ 0.12910 & 0.22351 & 0.19709 & 0.28129 & 0.05614 & 0.27284 \\ 0.26395 & 0.16693 & 0.16528 & 0.09218 & 0.19242 & 0.20010 \end{pmatrix}$$

$$\Omega_3 = \begin{pmatrix} 0.22732 & 0.17509 & 0.41652 \\ 0.24565 & 0.18920 & 0.45010 \\ 0.03812 & 0.02936 & 0.06985 \end{pmatrix}$$

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